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On a Special Case of Path Planning

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Abstract— **Spirals are curves of one sided, monotonically increasing or decreasing curvature. Spiral segments have the potential advantage that the minimum and maximum curvature exists at their end points. Moreover, spirals are free from inflection points and local curvature extrema. These properties make the study of spiral segments an interesting problem both in practical and aesthetic applications. An important issue in this study has been to design spirals which interpolate to given positional and tangential end conditions. Previous works suggest different methods for tackling this and similar problems. This paper aims to improve the existing methods for deriving C shaped spiral curves between two circles.**

I. Introduction

Planer spirals have monotonic curvature and are free from inflection points. Spirals have curvature extrema at their end points (no local curvature extrema), this makes spirals important for both physical [Gibreel et al.(1999)] and aesthetic applications [Burchard et al. (1993)] [1]. In curve and surface design, it is often desirable to have a spiral transition curve of G^2 contact. The purpose may be practical, e.g., in highway designs, railway routes, and in different CAD applications. This paper considers cubic spirals common to all modern design systems. Since then the derivative of the curvature is quintic, which means that it is not easy to locate their zeros, the following simplified problems have been considered in previous research papers [2][3][5][7] joining the following pairs of objects: (i) straight line to circle (J transition), (ii) circle to circle with a broken back C transition, (iii) circle to circle with an S transition, (iv) straight line to straight line (J to circle and circle to J) and (v) circle to circle with one circle inside the other. Walton and Meek have proposed the use of cubic and PH quintic splines to tackle the five cases (i)-(iv)and given their theoretical analysis [2][3], where the cubic curves with the zero curvatures at the one end point have been widely used. Dietz and Piper have numerically computed values to aid in adjusting the selection of control points for building cubic spiral interpolations without zero curvature (at any end point) restriction, and a table is developed which helps us in selecting the inner control points of a particular parametric cubic spiral to interpolate end conditions via matching given end positions and tangents. Another table is developed that matches tangent angles and curvatures at the end points of the cubic spiral [1]. It helps us form the spiral joining (v) circle to circle where one circle lies inside the other.

This paper considers cubic spirals common to all modern design systems. Since there is no closed form for the roots of a general quintic polynomial, we first transform the unit interval [0, 1] to [0, ∞) and next apply a sufficient condition for roots to derive as theoretically as possible the spiral conditions on the positions and tangents at the endpoints. Like [1] our method does not pose the zero curvature restriction at any end point. However instead of a table, this paper provides a numerical method to find the middle control points for constructing spirals conforming to given positional and tangential conditions. Section 3 gives illustrative numerical examples.

II. Background

A. *Notations and conventions*

The usual Cartesian co-ordinate system is presumed. Boldface is used for points and vectors, e.g.,

$$\boldsymbol{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}.$$

The Euclidean norm or length of a vector $\boldsymbol{a}$ is denoted by the notation

$$\|\boldsymbol{a}\| = \sqrt{a_x^2 + a_y^2},$$

and $\boldsymbol{a}\|\boldsymbol{b}$ means the vector $\boldsymbol{a}$ is parallel to vector $\boldsymbol{b}$. The positive angle of a vector $\boldsymbol{a}$ is the counter-clockwise angle from the vector $(1, 0)$ to $\boldsymbol{a}$. The derivative of a function f is denoted by f'. To aid concise writing of mathematical expressions, the symbol $\times$ is used to denote the signed z-component of the usual three-dimensional cross-product of two vectors in the xy plane, e.g.,

$$\boldsymbol{a} \times \boldsymbol{b} = a_x b_y - a_y b_x = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \sin\theta,$$

where θ is the counterclockwise angle from $\boldsymbol{a}$ to $\boldsymbol{b}$. $\boldsymbol{a} \cdot \boldsymbol{b}$ denotes the usual inner product.

The signed curvature of a parametric curve $\boldsymbol{P}(t)$ in the plane is

$$\kappa(t) = \frac{\boldsymbol{P}'(t) \times \boldsymbol{P}''(t)}{\|\boldsymbol{P}'(t)\|^3}, \tag{1}$$

when $\|\boldsymbol{P}'(t)\|$ is non-zero. Positive curvature has the center of curvature on the left as one traverses the curve in the direction of increasing parameter. For non-zero curvature, the radius of curvature, positive by convention, is $1/|\kappa(t)|$.

The derivative $\kappa'(t)$ of the curvature in (1) yields

$$\kappa'(t) = \frac{\phi(t)}{\|\boldsymbol{P}'(t)\|^5}, \tag{2}$$

where

$$\phi(t) = \|P'(t)\|^2 \frac{\mathrm{d}}{\mathrm{d}t}\{P'(t) \times P''(t)\} - 3\{P'(t) \times P''(t)\}\{P'(t) \cdot P''(t)\}. \quad (3)$$

The term 'spiral' refers to a curved line segment whose curvature varies monotonically with constant sign. Following two conditions on the curvature establish the spiral.

$$\kappa(t) \geq 0, \quad \kappa'(t) \geq 0 \quad (4)$$

A G^2 point of contact of two curves is a point where the two curves meet and where their unit tangent vectors and signed curvatures match.

According to Kneser's theorem any circle of curvature of a spiral arc contains every smaller circle of curvature of the arc in its interior and in its turn is contained in the interior of every circle of curvature of greater radius. So two distinct circles of curvature of a spiral arc never intersect and we cannot find the transition curve with a single spiral segment between two tangent circles or intersecting circles.

III. Description of Method

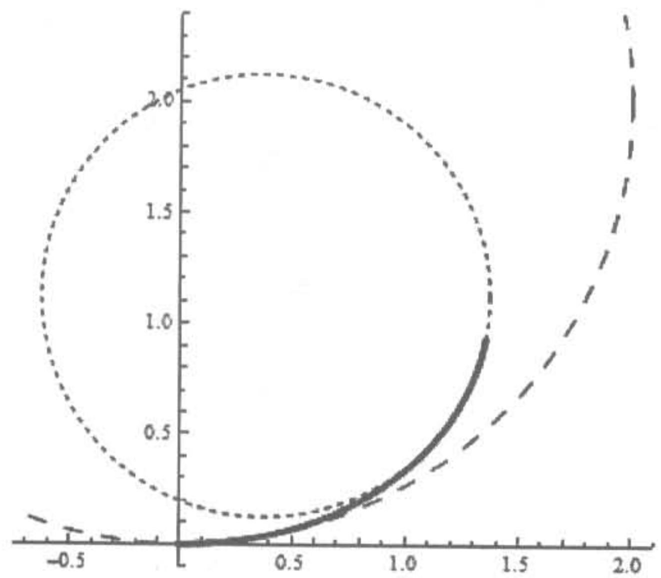

Fig. 1. A spiral segment between two circles.

Let $P_i, i = 0, \ldots, 3$ be four given control points. The cubic Bézier curve defined by them has eight degrees of freedom in the interval $0 \leq t \leq 1$ and is represented as

$$z(t) = (1-t)^3 p_0 + 3t(1-t)^2 p_1 + 3t^2(1-t)p_2 + t^3 p_3 \quad (5)$$

Its signed curvature $k(t)$is given by

$$\kappa(t) = \frac{z'(t) \times z''(t)}{\|z'(t)\|^3} \quad (6)$$

where "×", "."and $\|.\|$ mean the cross, inner products of two vectors and the Euclidean norm, respectively.

We assume throughout this paper that P_0 = (-1,0) and P_3 = (1,0). As in [2], the tangent lines for the parametric cubic at t = 0 and t = 1 (figure 2) and the horizonal axes for a triangle where the length of the lower two sides are $d_0 = 2sin\phi_1/sin(\phi_0+\phi_1)$ (for the side touching P_0) and $d_1 = 2sin\phi_0/sin(\phi_0+\phi_1)$ where $0 < \phi_0 < \phi_1 < \pi/2$. This condition restricts the curves to spiral arcs of increasing curvature [Guggenheimer (1963)]. As spirals may be parametrized and transformed (affine maps), this condition will not cause loss of generality. Parametrized spirals may be of increasing or decreasing curvature by changing the direction of parametrization. If both f_0 and f_1 are constrained to be between 0 and 1, parametric cubic spiral will be free from inflection points. For convexity, this and angle restriction is necessary [1].

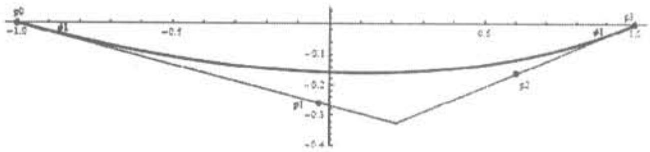

Fig. 2. Spiral arc of Figure 1 after transformation

Figure 1 shows a spiral segment between two circles (one containing the other) and Figure 2 shows its translated, rotated and scaled version.

The ratios (f_0, f_1) are extensively used in this paper:

$$f_0 = \frac{|p_1 - p_0|}{d_0}, f_1 = \frac{|p_2 - p_3|}{d_1} \quad (7)$$

to see

$$p_1 = p_0 + \frac{2f_0 sin\phi_1}{sin(\phi_0+\phi_1)}(cos\phi_0, -sin\phi_0), \quad (8)$$

$$p_2 = p_3 - \frac{2f_1 sin\phi_0}{sin(\phi_0+\phi_1)}(cos\phi_1, sin\phi_1) \quad (9)$$

To ensure that the parametric cubic curve (5) is free from inflection points, the ratios (f_0, f_1) are taken to be from zero to one.

Then, the curvatures at the endpoints are given

$$\kappa(0) = \frac{(1-f_1)sin\phi_0 sin^2(\phi_0+\phi_1)}{3f_0^2 sin^2\phi_1}, \quad (10)$$

$$\kappa(1) = \frac{(1-f_0)sin\phi_1 sin^2(\phi_0+\phi_1)}{3f_1^2 sin^2\phi_0}, \quad (11)$$

$$\|z'(t)\|^5 \kappa'(t) = \frac{2592 sin\phi_0 sin\phi_1}{(1+s)^5 sin^2(\phi_0+\phi_1)} \sum_{i=0}^{5} c_i s^i, t = 1/(1+s) \quad (12)$$

where

$c_0 = f_1^2 csc(\phi_0+\phi_1) sin(\phi_0)((3-3f_0^2-10f_1+9f_0f_1) sin\phi_0 + 3(-1+f_0)^2 sin(\phi_0)+2\phi_1))$

$c_1 = f_1(2(-1+f_0)(-6+6f_0^2+17f_1-21f_0f_1) cos\phi_1 sin\phi_0 + (-12f_0^3 + 22f_1^2 + 6f_0^2(2+7f_1) - 3f_0f_1(14 + 9f_1)) csc(\phi_0+\phi_1) sin^2\phi_0 - 12(-1+f_0)^3 cos\phi_0 sin\phi_1)$

$c_2 = 2f_1((-2+4f_1+f_0(5-32f_1+3f_0(4-5f_0+10f_1))) cos\phi_1 sin\phi_0 + (15f_0^3 - 2f_1^2 - 6f_0^2(2+5f_1) + 3f_0f_1(4+5f_1)) csc(\phi_0+\phi_1) sin^2\phi_0 + (-1+f_0)^2(-2+15f_0) cos\phi_0 sin\phi_1)$

$c_3 = 2f_0((2+6f_0(2-5f_1)f_1 + f_0^2(-2+15f_1) + f_1(-19+20f_1)) cos\phi_1 sin\phi_0 + (f_0^2(2-15f_1) + 3(4-5f_1)f_1^2 + 6f_0f_1(-2+5f_1)) csc(\phi_0+\phi_1) sin^2\phi_0 - (-1+f_0)(2-5f_1+f_0(-2+15f_1)) cos\phi_0 sin\phi_1)$

$c_4 = f_0((f_0^2(22-27f_1) + 42f_0(-1+f_1)f_1 - 12(-1+f_1)(-1+2f_1)) cos\phi_1 sin\phi_0 + (-42f_0(-1+f_1)f_1 + 12(-1+f_1)f_1^2 + f_0^2(-22+27f_1)) csc(\phi_0+\phi_1) sin^2\phi_0 + (12(-1+f_1) + f_0(34-22f_0-34f_1+27f_0f_1)) cos\phi_0 sin\phi_1)$

$c_5 = f_0^2 csc(\phi_0+\phi_1) sin\phi_1((-3+10f_0-9f_0f_1+3f_1^2) sin\phi_1 - 3(-1+f_1)^2 sin(2\phi_0+\phi_1))$

The parameter t has been replaced by t = 1/(1+s) to convert the domain of t from the stricter condition $0 \leq t \leq 1$ to $0 \leq s$. Now the following theorem establishes the spiral conditions:

Theorem 1: The transition cubic curve of the form (5) is a spiral for i = 0,1,..,4,5 if

$$c_i \geq 0 \tag{13}$$

IV. Examples

Figure 5 shows a spiral segment with starting point p_1 at (-1, 0) and ending point p_3 at (1, 0). The tangent angles of starting and ending tangents are $(\phi_0, \phi_1) = (\pi/12, \pi/8)$ where ϕ_0 and ϕ_1 conform to $0 \leq \phi_0 \leq \phi_1 \leq \pi/2$. In order to achieve the spiral segment of Figure 5 we first use Test (13) to find the possible values of f_0 and f_1 which allow a spiral segment. Figure 3 shows the plot of the spiral region of f_0 and f_1 for which these two ratios allow for circular arcs (The darker region is obtained from Test). Now using any value of f_0 and f_1 from the spiral region we can see that the curvature remains monotone and the resulting arcs are spiral. For example Figure 4 shows the curvature plot for $(f_0, f_1) = (0.8, 0.5)$ and Figure 5 shows the resulting Spiral. Similarly Figures 6 to 9 show the spiral regions, curvature plots and the resulting spirals for $(\phi_0, \phi1) = (\pi/10, \pi/8)$ and $(\phi_0, \phi1) = (\pi/4, \pi/3)$.
The Spiral regions are obtained using the RegionPlot[] command of Mathematica 6.0.

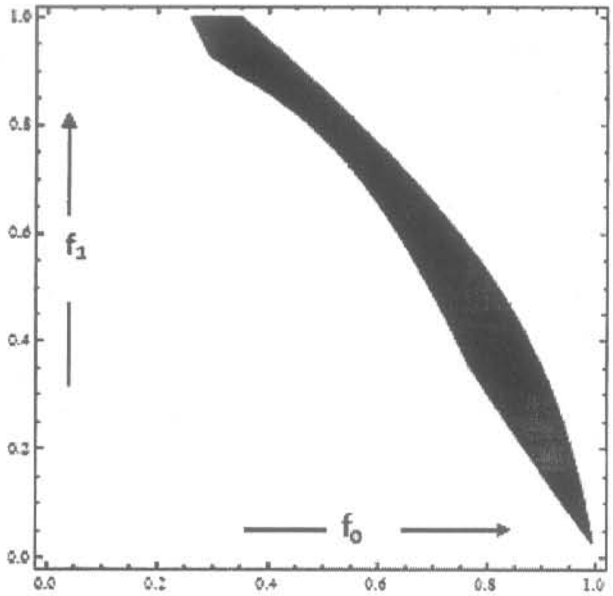

Fig. 3. Spiral region plotted between f_0(along x-axis) and f_1(along y-axis) for $(\phi_0, \phi_1)=(\pi/12, \pi/8)$

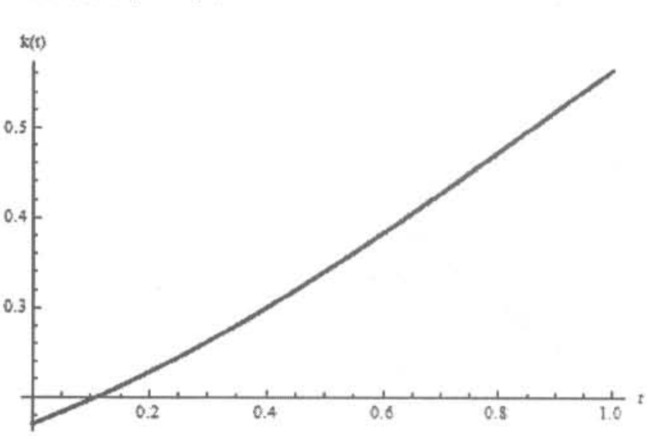

Fig. 4. Curvature plot for $(\phi_0, \phi_1)= (\pi/12, \pi/8)$and $(f_0, f_1)=(0.8, 0.5)$

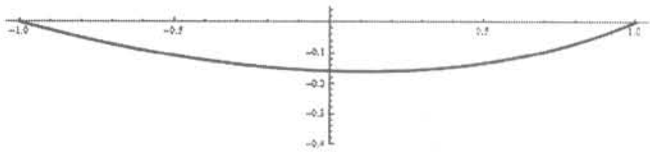

Fig. 5. Spiral curve between (-1,0) and (1,0)

V. Conclusion and Future Research

The work presented in this paper attempted to improve the existing methods for implementing spiral curves which conform to given spatial and tangential conditions by following a numerical approach. The condition formulated in Test (13) allows greater flexibility in spiral design than was allowed by the tabular approach developed in [1]. Moreover, formula based approaches suffer from lesser space complexity issues than tabular approaches in software implementations. Future

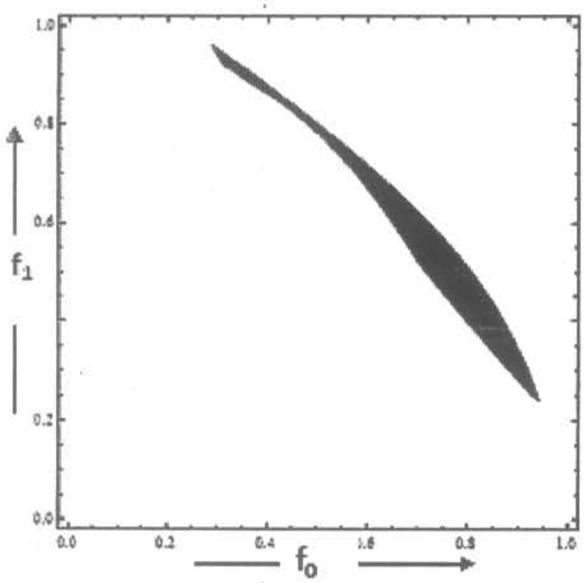

Fig. 6. Spiral region plotted between f_0(along x-axis) and f_1(along y-axis) for $(\phi_0, \phi_1)=(\pi/10, \pi/8)$

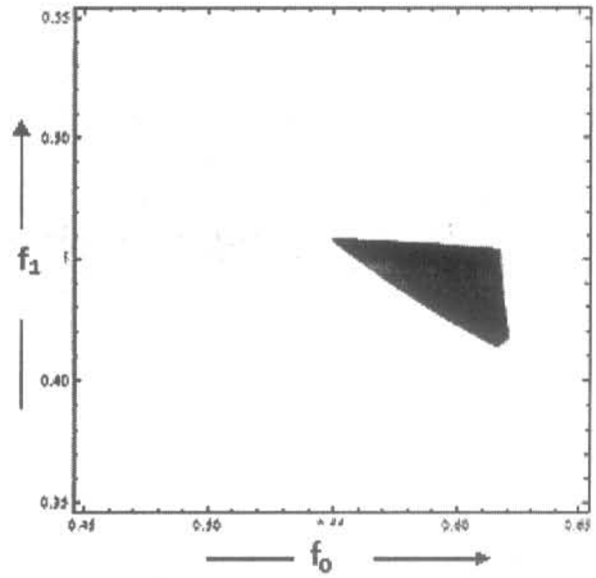

Fig. 9. Spiral region plotted between f_0(along x-axis) and f_1(along y-axis) for $(\phi_0, \phi_1)=(\pi/4, \pi/3)$

Fig. 7. Curvature plot for $(\phi_0, \phi_1)= (\pi/10, \pi/8)$and (f_0,f_1)=(0.8,0.5)

Fig. 8. Spiral curve between (-1,0) and (1,0)

work along this direction can be continued along various lines. For example the condition established by test is only sufficient condition and work can be done to find necessary condition for cubic Bezier spiral segments fitting spatial and tangential conditions. Secondly, the condition of the test can be further relaxed to allow for greater flexibility. Thirdly, in a broader perspective this paper just deals with spirals between circles with non-coinciding centers and work should be done to include concentric circles in the domain as well.

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